Sharp Thresholds for the Phase Transition between Primitive Recursive and Ackermannian Ramsey Numbers

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Abstract

We compute the sharp thresholds on g at which g-large and g-regressive Ramsey numbers cease to be primitive recursive and become Ackermannian.

We also identify the threshold below which g-regressive colorings have usual Ramsey numbers, that is, admit homogeneous, rather than just min-homogeneous sets.

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1 Introduction

Let $\mathbb{N}$ denote the set of all natural numbers including 0. A number $d \in \mathbb{N}$ is identified with the set $\{n \in \mathbb{N} : n < d\}$, and the set $\{0, 1, \ldots, d-1\}$ may also be sometimes denoted by $[d]$. The set of all *d-element subsets* of a set X is denoted by $[X]^d$. For a function $C : [X]^d \to \mathbb{N}$ we write $C(x_1, \ldots, x_d)$ for $C(\{x_1, \ldots, x_d\})$ under the assumption that $x_1 < \cdots < x_d$.

Definition 1.1 *A nonempty $H \subseteq \mathbb{N}$ is* g-large *for a function $g : \mathbb{N} \to \mathbb{N}$ if $|H| \geq g(\min H)$.*

The symbol

$$X \to_g^* (k)_c^d$$

means: for every coloring $C : [X]^d \to c$ there is a g-large C-homogeneous $H \subseteq X$ such that $|H| \geq k$. That is, the restriction of C to $[H]^d$ is a constant function.

In case $d = 2$, we just write $X \to_g^* (k)_c$.

Paris and Harrington [15] introduced the notion of a *relatively large* set of natural numbers, which is exactly g-large for $g = \mathrm{Id}$, and proved that the statement:

$$\mathrm{PH} \equiv (\forall d \geq 1, c > 0, k > 0)(\exists N)\ N \to_{\mathrm{Id}}^* (k)_c^d$$

is a Gödel sentence over Peano Arithmetic.

Fact 1.2 *Suppose $g : \mathbb{N} \to \mathbb{N}$ is any function. Then for every k, c and d there is some N so that $N \to_g^* (k)_c^d$.*

The proof follows from the infinite Ramsey theorem and compactness. See Paris and Harrington [15] for more details. □

The *g-large Ramsey number* of k and c, denoted $R_g^*(k, c)$, is the least N so that $N \to_g^* (k)_c$.

Erdős and Mills showed in their seminal paper [5] that R_{Id}^* is not primitive recursive. For a fixed number of colors the resulting Ramsey function is primitive recursive. When these Ramsey functions are considered as a hierarchy indexed by the number of colors then it is cofinal in the Grzegorczyk hierarchy of primitive recursive functions. Erdős and Mills further showed that the Ramsey function becomes double exponential if the number of colors is restricted to two.

Definition 1.3 *Given a set $X \subseteq \mathbb{N}$, a coloring $C : [X]^d \to \mathbb{N}$ is* g-regressive *for a function $g : \mathbb{N} \to \mathbb{N}$ if $C(x_1, \ldots, x_n) \leq g(x_1)$ for all $\{x_1, \ldots, x_n\} \subseteq X$.*

The symbol

$$X \overset{\min}{\to} (k)^d_g$$

means: for every g-regressive coloring $C : [X]^d \to \mathbb{N}$ there exists $H \subseteq X$ such that $|H| \geq k$ and H is min-homogeneous *for C, that is, $C(x, x_2, \ldots, x_d) = C(x, y_2, \ldots, y_d)$ for all $x, x_2, \ldots, x_d, y_2, \ldots, y_d \in H$.*

In case $d = 2$, we just write $X \overset{\min}{\to} (k)_g$.

Kanamori and McAloon [9] introduced the notion of a *g-regressive coloring* and proved that for $g = \mathrm{Id}$,

$$\mathrm{KM} \equiv (\forall d \geq 1, k > 0)(\exists N)\ N \overset{\min}{\to} (k)^d_{\mathrm{Id}}$$

is a Gödel sentence over Peano Arithmetic.

Fact 1.4 *Let $g : \mathbb{N} \to \mathbb{N}$ be arbitrary. Then*

(1) for every g-regressive coloring $C : [\mathbb{N}]^d \to \mathbb{N}$ there is an infinite $H \subseteq \mathbb{N}$ such that H is min-homogeneous for C.
(2) for any d and k there is some N so that for every g-regressive coloring $C : [N]^d \to \mathbb{N}$ there is a min-homogeneous $H \subseteq N$ of size at least k.

The first item follows from the infinite canonical Ramsey theorem, since a regressive coloring is equivalent neither to $\max\{x_1, \ldots, x_d\}$ nor to a 1-1 coloring on an infinite set. The second item follows from the first via compactness. See Kanamori and McAloon [9] for more details. □

The *g-regressive Ramsey number* of k, denoted $R^{\mathrm{reg}}_g(k)$, is the least N so that $N \overset{\min}{\to} (k)_g$.

Kanamori and McAloon also proved that $R^{\mathrm{reg}}_{\mathrm{Id}}$ is not primitive recursive. Purely combinatorial proofs of this can be found in [18] and in [11].

The symbol

$$X \to (k)_c$$

means that the standard Ramsey relation for pairs holds. Namely, for every coloring $C : [X]^2 \to c$ there is a C-homogeneous $H \subseteq X$ of size k.

Let $R(k, c)$ denote the least N such that $N \to (k)_c$ and let $R^{\min}(k, c)$ denote the least N so that, given a coloring $C\colon [N]^2 \to c$, there is some $H \in [N]^k$ which is min-homogeneous for C. Note that $R^{\min}(k, 1) = k$ and $R^{\min}(2, c) = 2$.

Recall that the standard proof of the finite Ramsey theorem gives, for $c, k \geq 2$:

$$c^k \overset{\min}{\to} (k)_c \quad \text{and} \quad c^{k \cdot c} \to (k)_c$$

That is, $R(k, c) \leq c^{k \cdot c}$ and $R^{\min}(k, c) \leq c^k$ for any $c, k \geq 2$.

For any function $f : \mathbb{N} \to \mathbb{N}$ the function $f^{(n)}$ is defined by $f^{(1)}(x) = x + 1$ and $f^{(n+1)}(x) = f(f^{(n)}(x))$.

Definition 1.5 *The Ackermann function is defined as* $\mathrm{Ack}(n) = A_n(n)$ *for all* $n > 0$ *(and, say,* $\mathrm{Ack}(0) = 0$*) where each* A_n *is the standard* n-th approximation *of the Ackermann function, defined by:*

$$
\begin{aligned}
A_1(n) &= n + 1 \\
A_{i+1}(n) &= A_i^{(n)}(n)
\end{aligned}
$$

Let us record that $\mathrm{Ack}(1) = 2$, $\mathrm{Ack}(2) = 4$, $\mathrm{Ack}(3) = 24$, $2^{2^{2^{70}}} < \mathrm{Ack}(4) < 2^{2^{2^{71}}}$ for later use.

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, g *eventually dominates* or *grows eventually faster than* f if there is some N so that for all $i \geq N$ it holds that $f(i) \leq g(i)$. In that case we also say that f is *eventually dominated* by g. We call f *nondecreasing* if for any $i < j$ we have $f(i) \leq f(j)$. A function $h : \mathbb{N} \to \mathbb{N}$ is *unbounded* if for every $N \in \mathbb{N}$ there exists an i such that $h(i) > N$.

The class of *primitive recursive functions* is the smallest class of functions from $\mathbb{N}^d$ to $\mathbb{N}$ for all $d \geq 1$ which contains the constant functions, the projections, and the successor function and is closed under composition and recursion. This class is also closed under *bounded search*, frequently referred to as *bounded μ-operator*. See e.g. [3, 17] for more details about the class of primitive recursive functions.

It is well known (see e.g. [3]) that each approximation A_n is *primitive recursive* and that every primitive recursive function is eventually dominated by some A_n. Thus the Ackermann function eventually dominates every primitive recursive function.

Definition 1.6 *A function* $g : \mathbb{N} \to \mathbb{N}$ *is said to be* Ackermannian *if it grows eventually faster than every primitive recursive function.*

There is no *smallest* Ackermannian function: if f is Ackermannian, then so is $i \mapsto f(i)/2$ or $i \mapsto f(i)^{1/2}$, etc. It is also important to note that there are functions $f : \mathbb{N} \to \mathbb{N}$ which are neither Ackermannian nor eventually dominated by any primitive recursive function.

Lemma 1.7 *If the composition* $f \circ g$ *of two nondecreasing functions is Ackermannian and one of* f *and* g *is primitive recursive, then the other is Ackermannian.*

PROOF. If f is primitive recursive, then g should be Ackermannian. Assume now g is primitive recursive. Note that g is not bounded. And, given a primitive recursive function p, the function $h(n) := p(g(n+1))$ is primitive recursive too, so there is some N such that $f(g(n)) \geq h(n) = p(g(n+1))$ for all $n \geq N$. Since we can assume w.l.o.g. that p is nondecreasing, it holds for all $i \geq g(N)$ that $f(i) \geq f(g(n)) \geq p(g(n+1)) \geq p(i)$, where $g(n) \leq i \leq g(n+1)$ for some $n \geq N$. Hence f is Ackermannian. $\square$

We compute below the sharp thresholds on g at which g-large and g-regressive Ramsey numbers cease to be primitive recursive and become Ackermannian. We prove:

Theorem A. *Suppose $g : \mathbb{N} \to \mathbb{N}$ is nondecreasing and unbounded. Then R_g^* is eventually dominated by some primitive recursive function if and only if for every $t > 0$ there is some $M(t)$ so that for all $n \geq M(t)$ it holds that*

$$g(n) < \frac{\log(n)}{t}$$

and $M(t)$ is primitive recursive in t.

Here log denotes the logarithm to base 2.

Theorem B. *Suppose $g : \mathbb{N} \to \mathbb{N}$ is nondecreasing and unbounded. Then R_g^{reg} is bounded by some primitive recursive function if and only if for every $t > 0$ there is some $M(t)$ so that for all $n \geq M(t)$ it holds that*

$$g(n) < n^{1/t}$$

and $M(t)$ is primitive recursive in t.

We also identify the threshold below which g-regressive colorings have usual Ramsey numbers, that is, admit homogeneous, rather than just min-homogeneous sets, and give a lower bound of $A_{53}(2^{2^{274}})$ on the Id-regressive Ramsey number of $k = 82$, where A_{53} is the 53th approximation of Ackermann's function.

For an unbounded and nondecreasing function $g : \mathbb{N} \to \mathbb{N}$ define the inverse function $g^{-1} : \mathbb{N} \to \mathbb{N}$ by

$$g^{-1}(m) := \begin{cases} \ell & \text{if } \ell := \min\{i : g(i) \geq m\} > 0\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Let us remark that although Ack is not primitive recursive, its inverse Ack^{-1} is primitive recursive.

2 The Phase Transition of g-regressive Ramsey numbers.

We now show that the threshold for Ackermannian g-regressive Ramsey numbers lies above all functions $n^{1/f^{-1}(n)}$ obtained from a primitive recursive f and below $n^{1/\operatorname{Ack}^{-1}(n)}$.

Worded differently, for a nondecreasing and unbounded g to have primitive recursive g-regressive Ramsey numbers it is necessary and sufficient that g is eventually dominated by $n^{1/t}$ for all $t > 0$ and that the rate at which g gets below $n^{1/t}$ is not too slow: if g gets below $n^{1/t}$ only after an Ackermannianly long time M_t, then the g-regressive Ramsey numbers are still Ackermannian.

We begin with the following lemma which stems from Lemma 26.4 in [4].

Lemma 2.1 $R^{\min}(k,c) \leq 2 \cdot c^{k-2}$ *for any* c, $k \geq 2$.

Note that Lemma 26.4 in [4] talks about *end-homogeneous* sets. However, if we confine ourselves to the 2-dimensional case it is just about min-homogeneous sets. Concerning n-dimensional min-homogeneous sets see [12].

Theorem 2.2 *Given* $B : \mathbb{N} \to \mathbb{N}^+$ *let* $g_B(i) := \lfloor i^{1/B^{-1}(i)} \rfloor$. *Assume* B *is nondecreasing and unbounded. Then for every* $k \geq 2$ *such that* $B(k^2) \geq 2$ *it holds that* $(B(k^2))^{k+1} \overset{\min}{\longrightarrow} (k)_{g_B}$.

PROOF. Given $k \geq 2$ such that $B(k^2) \geq 2$ set

$$N := (B(k^2))^{k+1} \quad \text{and} \quad \ell := 2 \cdot (B(k^2))^k \leq N\,.$$

Now let $C\colon [N]^2 \to \mathbb{N}$ be a g_B-regressive function. Consider the function $D\colon [B(k^2), \ell]^2 \to \mathbb{N}$ defined from C by restriction. For any $y \in [B(k^2), \ell]$ we have

$$y^{\frac{1}{B^{-1}(y)}} \leq (B(k^2))^{\frac{k+1}{B^{-1}(B(k^2))}} = (B(k^2))^{(k+1)\cdot k^{-2}}$$

which implies that $\operatorname{Im}(D) \subseteq (B(k^2))^{(k+1)\cdot k^{-2}} + 1$. On the other hand,

$$2 \cdot ((B(k^2))^{(k+1)\cdot k^{-2}} + 1)^{k-2} < ((B(k^2))^{(k+1)\cdot k^{-2}+1})^{k-1} < (B(k^2))^k.$$

By Lemma 2.1 there is some k-element set H which is min-homogeneous for D, and hence for C. □

Corollary 2.3 *Suppose* $B : \mathbb{N} \to \mathbb{N}^+$ *is unbounded, nondecreasing and* $g(n) \leq g_B(n) = \lfloor n^{1/B^{-1}(n)} \rfloor$ *for all* n *. If* B *is bounded by a primitive recursive function, then* R_g^{reg} *is bounded by a primitive recursive function. If, in addition,* g *itself is primitive recursive, then* R_g^{reg} *is primitive recursive.*

PROOF. By the theorem above R_g^{reg} is eventually dominated by $(B(k^2))^{k+1}$ and thus is bounded by a primitive recursive function. If, in addition, g is primitive recursive, then the relation $N \xrightarrow{\min} (k)_g$ is a primitive recursive relation and the computation of R_g^{reg} requires only a bounded search for a solution for a primitive recursive relation and therefore R_g^{reg} is primitive recursive. $\square$

We provide now two different proofs for the upper threshold, by displaying two different "bad" colorings, each based on a different combinatorial proof of the fact the Id-regressive Ramsey numbers are Ackermannian [11, 18]. The first proof makes use of the idea from [18], and the second proof uses the idea of [11]. Both colorings are based on the idea of expanding the difference between two natural numbers by a "moving" base, depending on the position of the pair.

The first bad coloring we give codes "half" of the information that the second coloring codes: the color of $\{m, n\}$ according to the first coloring is the first different digit in the expansions of m and n, whereas according to the second it is the pair consisting of that digit and its position. The missing information in the first coloring is compensated by composing the regressive Ramsey function with the usual Ramsey function. The first proof is essentially asymptotic.

In the second proof we construct a single, simply computable $n^{1/\operatorname{Ack}^{-1}(n)}$-regressive, primitive recursive coloring of $[\mathbb{N}]^2$. It requires more detailed analysis of variants of approximations of Ackermann's function, but in return the result is less asymptotic and enables estimates of $R_{\text{Id}}^{\text{reg}}(k)$ for relatively small values of k. For instance, we show that $R_{\text{Id}}^{\text{reg}}(82)$ is larger than $A_{53}(2^{2^{274}})$.

2.1 g-regressive upper threshold – first proof

We now begin working towards the first proof of the converse of Corollary 2.3: if f^{-1} is Ackermannian and $g(n) = n^{1/f(n)}$ then R_g^{reg} is Ackermannian. This proof generalizes the method developed in [18] and [11].

Definition 2.4 *For a given $t \in \mathbb{N} \setminus \{0\}$, we define a sequence of functions $(f_t)_i : \mathbb{N} \to \mathbb{N}$ as follows.*

$$(f_t)_1(n) = n + 1$$
$$(f_t)_{i+1}(n) = (f_t)_i^{(\lfloor n^{1/t} \rfloor)}(n)$$

Note that $(f_t)_i$ are strictly increasing. We would first like to show that the function $k \mapsto (f_t)_k(k)$ is Ackermannian for all $t > 0$. To do that, we will use the following claims.

Claim 2.5 *For every $t, k, n > 0$ it holds that $(f_t)_k(n) \geq n + (\lfloor n^{1/t} \rfloor)^{k-1}$*

PROOF. We show the claim by induction on k. If $k = 1$, it follows by definition that $(f_t)_k(n) = n + 1 = n + (\lfloor n^{1/t} \rfloor)^{k-1}$. Let $k \geq 1$. By definition $(f_t)_{k+1}(n) = (f_t)_k^{(\lfloor n^{1/t} \rfloor)}(n)$ and by applying the induction hypothesis $\lfloor n^{1/t} \rfloor$ times we get that the right hand side of the equation is larger than $n + ((\lfloor n^{1/t} \rfloor)(\lfloor n^{1/t} \rfloor)^{k-1})$ which is $n + (\lfloor n^{1/t} \rfloor)^k$. $\square$

Claim 2.6 *For every $t, k > 0$ and $n > 2^{t+1}$ it holds that $(f_{t+1})_{2t+3}(n^2) > n^2 + 2n + 1$*

PROOF. By Claim 2.5 we have that $(f_{t+1})_{2t+3}(n^2) \geq n^2 + (\lfloor n^{\frac{2}{t+1}} \rfloor)^{2t+2}$. Now

$$\begin{aligned}
n^2 + (\lfloor n^{\frac{2}{t+1}} \rfloor)^{2t+2} &\geq n^2 + (n^{\frac{2}{t+1}} - 1)^{2(t+1)} \\
&\geq n^2 + (n^{\frac{4}{t+1}} - 2n^{\frac{2}{t+1}} + 1)^{t+1} \\
&> n^2 + (n^{\frac{2}{t+1}}(n^{\frac{2}{t+1}} - 2))^{t+1} \\
&> 2n^2 \\
&> n^2 + 2n + 1
\end{aligned}$$

for any $t, k > 0$ and $n > 2^{t+1}$. $\square$

Claim 2.7 *Let $t > 0$. For all $n > 2^{t+1}$, $i > 0$ it holds that*

$$(f_{t+1})_{i+2t+2}(n^2) > ((f_t)_i(n))^2$$

PROOF. We prove the claim by induction on i. For $i = 1$, by Claim 2.6,

$$(f_{t+1})_{i+2t+2}(n^2) = (f_{t+1})_{2t+3}(n^2) > n^2 + 2n + 1 = ((f_t)_1(n))^2 = ((f_t)_i(n))^2 .$$

We now assume that Claim 2.7 is true for i and prove it for $i + 1$. To do that we need the following claim:

Claim 2.8 *For any $j \in \mathbb{N}^+$ it holds that $(f_{t+1})_{i+2t+2}^{(j)}(n^2) > ((f_t)_i^{(j)}(n))^2$.*

PROOF. We show Claim 2.8 by induction on j. For $j = 1$ the claim can be shown by a simple induction on i. For $j > 1$ we have

$$(f_{t+1})_{i+2t+2}^{(j+1)}(n^2) = (f_{t+1})_{i+2t+2}((f_{t+1})_{i+2t+2}^{(j)}(n^2)).$$

The latter term is larger than $(f_{t+1})_{i+2t+2}(((f_t)_i^{(j)}(n))^2)$ by monotonicity and the induction hypothesis for j. Now, if we denote $n' = (f_t)_i^{(j)}(n)$, we easily see, by the induction hypothesis for the case $j = 1$, that

$$(f_{t+1})_{i+2t+2}(((f_t)_i^{(j)}(n))^2) > ((f_t)_i((f_t)_i^{(j)}(n)))^2$$

which is, in fact, $((f_t)_i^{(j+1)}(n))^2$. □

We still need to show the induction step for Claim 2.7. We have

$$(f_{t+1})_{i+1+2t+2}(n^2) = (f_{t+1})_{i+2t+2}^{(\lfloor n^{\frac{2}{t+1}} \rfloor)}(n^2) \geq (f_{t+1})_{i+2t+2}^{(\lfloor n^{1/t} \rfloor)}(n^2)\,.$$

By Claim 2.8, the latter term is larger than $((f_t)_i^{(\lfloor n^{1/t} \rfloor)}(n))^2 = ((f_t)_{i+1}(n))^2$.□

Claim 2.9 *For all $t > 0$, $n > 4$ it holds that $(f_{t+1})_{i+t^2+3t}(n^{2^t}) > (A_i(n))^{2^t}$.*

PROOF. Observe that $(A_i(n))^{2^t}$ is actually $((f_1)_i(n))^{2^t}$. Now, by applying claim 2.7 to the latter term, we get $((f_1)_i(n))^{2^t} < ((f_2)_{i+2+2}(n^2))^{2^{t-1}}$, since the parameter t of claim 2.7 is 1 here. If we apply it now to the right hand side term, the parameter t of the claim would be 2 and we would find that this term is smaller than $((f_3)_{i+2+2+4+2}(n^2))^{2^{t-1}}$. Generally, if we apply the claim j times we get that $((f_1)_i(n))^{2^t} < ((f_{j+1})_{i+j^2+3j}(n^{2^j}))^{2^{t-j}}$ since we may replace $\sum_{l=1}^{j} 2j$ with $j^2 + j$. Thus, if we let $j = t$, we get the desired inequality. Note that we are allowed to apply claim 2.7 t times, only if, for all $1 \leq j \leq t$ it holds that $n^{2^{j-1}} > 2^{j+1}$, which holds for every $n > 4$. □

Claim 2.10 *For every $t > 0$ and $n > 3^t$ it holds that $(f_t)_{4t+1}(n) > n^2$.*

PROOF. Applying Claim 2.5 with $k = 4t + 1$ we have $(f_t)_{4t+1}(n) \geq n + (\lfloor n^{1/t} \rfloor)^{4t}$ and the latter term is larger than $((n^{1/t}-1)^2)^{2t}$ which equals $((n^{2/t} - 2n^{1/t} + 1))^{2t} > (n^{1/t}(n^{1/t} - 2))^{2t}$. Now, since $n > 3^t$ we know that $n^{1/t} - 2 > 1$ and thus, the latter term is larger than $(n^{1/t})^{2t} = n^2$. □

Claim 2.11 *For every $t > 0$ and $n > \max\{3^t, t^t\}$ it holds that $(f_t)_{4t+2}(n) > n^{2^t}$.*

PROOF. By definition $(f_t)_{4t+2}(n) = (f_t)_{4t+1}^{(\lfloor n^{1/t} \rfloor)}(n)$ which is not less than $(f_t)_{4t+1}^{(t)}(n)$ since $n > t^t$. Now, applying claim 2.10 t times, we get $(f_t)_{4t+1}^{(t)}(n) > n^{2^t}$ since f_t is monotone. □

Claim 2.12 *For any $t > 0$ and $n > \max\{4, 3^{t+1}, (t+1)^{t+1}\}$ it holds that $(f_{t+1})_{i+t^2+4t+5}(n) > A_i(n)$ for any $i > 0$.*

PROOF. Since $n > 2^{t+1}$, we have that

$$(f_{t+1})_{i+t^2+4t+5}(n) = (f_{t+1})^{(\lfloor n^{1/(t+1)} \rfloor)}_{i+t^2+4t+4}(n) > (f_{t+1})^{(2)}_{i+t^2+4t+4}(n).$$

The latter term is clearly larger than $(f_{t+1})_{i+t^2+3t}((f_{t+1})_{4t+6}(n))$ since $i, t > 0$. By claim 2.11 we have $(f_{t+1})_{4t+6}(n) > n^{2^{t+1}}$ and thus, by claim 2.9 we get

$$(f_{t+1})_{i+t^2+3t}((f_{t+1})_{4t+6}(n)) > (f_{t+1})_{i+t^2+3t}(n^{2^t}) > (A_i(n))^{2^t}$$

which is clearly larger than $A_i(n)$. □

We are now ready to establish that the growth rate of $k \mapsto (f_t)_k(k)$ is Ackermannian in terms of k. It is important to observe that this says more than just the fact that every primitive recursive function is eventually dominated by $(f_t)_i$ for some i.

Claim 2.13 *For all $0 < t \in \mathbb{N}$ the function $k \mapsto (f_t)_k(k)$ is Ackermannian.*

PROOF. For $t = 1$ the functions $(f_t)_k = A_k$, the standard k-th approximations of Ackermann's functions, so every primitive recursive function is eventually dominated by $f_t(k)$ (see e.g. [3]).

For $t > 1$ It suffices to show that for every $i \in \mathbb{N}$, the function $(f_t)_k(k)$ eventually dominates $A_i(k)$. Namely, that there exists $m_i \in \mathbb{N}$ such that for every $m > m_i$ it holds that $(f_t)_m(m) > A_i(m)$. But if we set $m_i = \max(\{(t+1)^{t+1}, i + t^2 + 4t + 5\})$, then by claim 2.12 we get exactly that since for any $m > m_i$ it holds that $(f_t)_m(m) > (f_{t+1})_{i+t^2+4t+5}(m) > A_i(m)$. □

We now turn to the converse of Corollary 2.3.

Definition 2.14 *Given $t \in \mathbb{N}^+$ set,*

$$g_t(n) := \lfloor n^{1/t} \rfloor .$$

Lemma 2.15 *$R^{\text{reg}}_{g_t}(R(n+3, c)) \geq (f_t)_{c+1}(n)$ for any c and n.*

PROOF. Let $k := R(n+3, c)$ and define a function $C_t \colon [R^{\text{reg}}_{g_t}(k)]^2 \to \mathbb{N}$ as follows:

$$C_t(x, y) := \begin{cases} 0 & \text{if } (f_t)_{c+1}(x) \leq y, \\ \ell & \text{otherwise,} \end{cases}$$

where the number ℓ is defined by

$$(f_t)^{(\ell)}_p(x) \leq y < (f_t)^{(\ell+1)}_p(x)$$

where $0 < p = \max\{q\colon (f_t)_q(x) \le y\} < c+1$. Note that C_t is g_t-regressive since $(f_t)_p^{(\lfloor x^{1/t}\rfloor)}(x) = (f_t)_{p+1}(x)$. Let H be a k-element subset of $R^{\mathrm{reg}}_{g_t}(k)$ which is min-homogeneous for C_t. Define a c-coloring $D_t\colon [H]^2 \to c$ by

$$D_t(x,y) := \begin{cases} 0 & \text{if } (f_t)_{c+1}(x) \le y, \\ p-1 & \text{otherwise,} \end{cases}$$

where p is as above. Then there is an $(n+3)$-element set $Y \subseteq H$ homogeneous for D_t. Let $x < y < z$ be the last three elements of Y. Then $n \le x$ and thus it suffices to show that $(f_t)_{c+1}(x) \le y$ since $(f_t)_{c+1}$ is an increasing function.

Assume $(f_t)_{c+1}(x) > y$. Then $(f_t)_{c+1}(y) \ge (f_t)_{c+1}(x) > z$ by the min-homogeneity. Let $C_t(x,y) = C_t(x,z) = \ell$ and $D_t(x,y) = D_t(x,z) = D_t(y,z) = p-1$. Then

$$(f_t)_p^{(\ell)}(x) \le y < z < (f_t)_p^{(\ell+1)}(x).$$

This implies that $z < (f_t)_p^{(\ell+1)}(x) \le (f_t)_p(y) \le z$. Contradiction! $\square$

Corollary 2.16 *$R^{\mathrm{reg}}_{g_t}$ is Ackermannian for any $t \in \mathbb{N}^+$.*

PROOF. It is obvious by Claim 2.13 since $R^{\mathrm{reg}}_{g_t}$ is nondecreasing. $\square$

Theorem 2.17 *Suppose $f : \mathbb{N} \to \mathbb{N}$ is nonzero, nondecreasing and unbounded, and $f(i) \le \mathrm{Ack}(i)$ for all i. Let $g(i) := \lfloor i^{1/f^{-1}(i)} \rfloor$. Then*

$$R^{\mathrm{reg}}_g(R(4 + 3^{i+1} + (i+1)^{i+1} + 3, i + i^2 + 4i + 5)) > f(i+1)$$

for all i.

PROOF. Let $p(i) := 4 + 3^{i+1} + (i+1)^{i+1}$ and $q(i) := i + i^2 + 4i + 5$. Assume to the contrary that for some i

$$N(i) := R^{\mathrm{reg}}_g(R(p(i)+3, q(i))) \le f(i+1)\,.$$

For all $\ell \le N(i)$ we have $f^{-1}(\ell) \le i+1$, and hence $\ell^{1/(i+1)} \le \ell^{1/(f^{-1}(\ell))}$. Then

$$\begin{aligned} R^{\mathrm{reg}}_g(R(p(i)+3, q(i))) &\ge R^{\mathrm{reg}}_{g_{i+1}}(R(p(i)+3, q(i))) \\ &\ge (f_{i+1})_{q(i)+1}(p(i)) \\ &> A_{i+1}(p(i)) \\ &\ge \mathrm{Ack}(i+1) \\ &\ge f(i+1) \end{aligned}$$

by Lemma 2.15 and Claim 2.12. Contradiction! $\square$

Theorem 2.18 *Suppose $B : \mathbb{N} \to \mathbb{N}$ is positive, unbounded and nondecreasing. Let $g_B(i) := \lfloor i^{1/B^{-1}(i)} \rfloor$. Then $R^{\text{reg}}_{g_B}(k)$ is Ackermannian iff B is Ackermannian.*

PROOF. Suppose B is Ackermannian. By replacing B with $\min\{B, \text{Ack}\}$, we assume that $B(i) \leq \text{Ack}(i)$ for all $i \in \mathbb{N}$. That $R^{\text{reg}}_{g_B}$ is Ackermannian follows from the previous theorem, since $r(i) := R(4+3^{i+1}+(i+1)^{i+1}+3, i+i^2+4i+5)$ is primitive recursive.

Suppose now that B is not Ackermannian, and fix an increasing primitive recursive function f so that for infinitely many $i \in \mathbb{N}$ it holds that $B(i) < f(i)$. On the other hand, it holds by Theorem 2.2 that $R^{\text{reg}}_{g_B}(k) \leq (B(k^2))^{k+1}$ for any $k \geq 2$ such that $B(k^2) \geq 2$. Hence it holds that $R^{\text{reg}}_{g_B}(i) \leq (f(i^2))^{i+1}$ for infinitely many $i \in \mathbb{N}$. This means that, for infinitely many $i \in \mathbb{N}$, $R^{\text{reg}}_{g_B}(i)$ is bounded by $f'(i)$ for some primitive recursive $f' : \mathbb{N} \to \mathbb{N}$. □

2.2 g-regressive upper threshold – second proof

We now begin the second proof by presenting a general method for constructing a "bad" g-regressive coloring which is a generalization of the method from [11]. In other words, given a function g and a natural number k, we present a g-regressive coloring C_g of pairs over a segment of size depending on g and k such that there is no min-homogeneous set for C_g of size $k+1$ within that segment. We then further show that if $g(n) = n^{1/r}$ for $r > 0$, then the size of the segment we may color is Ackermannian in terms of k. We then use this general coloring method to construct a single $n^{1/\text{Ack}^{-1}(n)}$-regressive "bad" coloring of $[\mathbb{N}]^2$.

Let $k > 2$ be a natural number and $g : \mathbb{N} \to \mathbb{N}$ a nondecreasing function such that for some $t \in \mathbb{N}$ it holds that $k \leq \frac{\sqrt{g(t)}}{2}$. Let $\mu_g : \mathbb{N} \to \mathbb{N}$ be a function which satisfies for all $k \in \mathbb{N}$ that $k \leq \frac{\sqrt{g(\mu_g(k))}}{2}$.

Definition 2.19 *We define a sequence of functions $(f_g)_i : \mathbb{N} \to \mathbb{N}$ as follows.*

$$(f_g)_1(n) = n + 1$$
$$(f_g)_{i+1}(n) = (f_g)_i^{(\lfloor \frac{\sqrt{g(n)}}{2} \rfloor)}(n)$$

Define a sequence of semi-metrics $\langle (d_g)_i : i \in \mathbb{N} \rangle$ on $\{n : n \geq \mu_g(k)\}$ by setting, for $m, n \geq \mu_g(k)$,

$$(d_g)_i(m, n) = |\{\ell \in \mathbb{N} : m < (f_g)_i^{(\ell)}(\mu_g(k)) \leq n\}|$$

For $n > m \geq \mu_g(k)$ let $I_g(m,n)$ be the greatest i for which $(d_g)_i(m,n)$ is positive, and $D_g(m,n) = (d_g)_{I(m,n)}(m,n)$.

Let us fix the following (standard) pairing function Pr on $\mathbb{N}^2$:

$$\Pr(m,n) = \binom{m+n+1}{2} + n$$

$\Pr : \mathbb{N}^2 \to \mathbb{N}$ is bijective and monotone in each variable. Observe that if $m, n \leq \ell$ then $\Pr(m,n) < 4\ell^2$ for all $\ell > 2$.

Definition 2.20 *Given a natural number $k > 2$ and a nondecreasing function $g : \mathbb{N} \to \mathbb{N}$, we define a pair coloring C_g on $[\{n : n \geq \mu_g(k)\}]^2$ as follows:*

$$C_g(m,n) = \Pr(I_g(m,n), D_g(m,n))$$

Claim 2.21 $D_g(m,n) \leq \frac{\sqrt{g(m)}}{2}$ *for all $n > m \geq \mu_g(k)$.*

PROOF. Let $i = I_g(m,n)$. Since $(d_g)_{i+1}(m,n) = 0$, there exist t and ℓ such that

$$t = (f_g)_{i+1}^{(\ell)}(\mu_g(k)) \leq m < n < (f_g)_{i+1}^{(\ell+1)}(\mu_g(k)) = (f_g)_{i+1}(t).$$

But $(f_g)_{i+1}(t) = (f_g)_i^{(\lfloor \frac{\sqrt{g(t)}}{2} \rfloor)}(t)$ and therefore $\frac{\sqrt{g(t)}}{2} \geq (d_g)_i(t, (f_g)_{i+1}(t)) \geq D_g(m,n)$. □

Claim 2.22 *C_g is g-regressive on the interval $[\mu_g(k), (f_g)_k(\mu_g(k)))$.*

PROOF. Clearly, $(d_g)_k(m,n) = 0$ for $\mu_g(k) \leq m < n < (f_g)_k(\mu_g(k))$ and therefore $I_g(m,n) < k \leq \frac{\sqrt{g(m)}}{2}$. From Claim 2.21 we know $D_g(m,n) \leq \frac{\sqrt{g(m)}}{2}$. Thus, $C_g(\{m,n\}) \leq \Pr(\lfloor \frac{\sqrt{g(m)}}{2} \rfloor, \lfloor \frac{\sqrt{g(m)}}{2} \rfloor)$, which is $< g(m)$ since $\frac{\sqrt{g(m)}}{2} > 2$. □

Claim 2.23 *For every $i \in \mathbb{N}$, every sequence $x_0 < x_1 < \cdots < x_i$ that satisfies $(d_g)_i(x_0, x_i) = 0$ is not min-homogeneous for C_g.*

PROOF. The claim is proved by induction on i. If $i = 1$ then there are no $x_0 < x_1$ with $(d_g)_1(x_0, x_1) = 0$ at all. Let $i > 1$ and suppose to the contrary that $x_0 < x_1 < \cdots < x_i$ form a min-homogeneous sequence with respect to C_g and that $(d_g)_i(x_0, x_i) = 0$. Necessarily, $I_g(x_0, x_i) = j < i$. By min-homogeneity, $I(x_0, x_1) = j$ as well, and $(d_g)_j(x_0, x_i) = (d_g)_j(x_0, x_1)$. Hence, $\{x_1, x_2, \ldots x_i\}$ is min-homogeneous with $(d_g)_j(x_1, x_i) = 0$ – contrary to the induction hypothesis. □

Corollary 2.24 *There exists no $H \subseteq [\mu_g(k), (f_g)_k(\mu_g(k)))$ of size $k+1$ that is min-homogeneous for C_g.*

Corollary 2.25 *Assume that the function $k \mapsto (f_g)_k(k)$ is Ackermannian. If there exists a function μ_g that is bounded by some primitive recursive function and satisfies for all k that $k \leq \mu_g(k)$ and that $k \leq \frac{\sqrt{g(\mu_g(k))}}{2}$, then R_g^{reg} is also Ackermannian.*

PROOF. First consider the function $C_g' : [(f_g)_k(\mu_g(k))]^2 \to \mathbb{N}$ defined by

$$C_g'(m,n) := \begin{cases} 0 & \text{if } m < \mu_g(k), \\ C_g(m,n) & \text{otherwise.} \end{cases}$$

Note that C_g' is g-regressive and has, by Corollary 2.24, no min-homogeneous set of size $\mu_g(k)+k+1$. Hence, we have $R_g^{\mathrm{reg}}(\mu_g(k)+k+1) > (f_g)_k(\mu_g(k))$.

On the other hand, the function $k \mapsto (f_g)_k(\mu_g(k))$ is obviously Ackermannian. Therefore, R_g^{reg} is also Ackermannian because $\mu_g(k)$ is bounded by some primitive recursive function. (See the proof of Lemma 1.7). □

Lemma 2.26 *Given a real number $r > 0$ let $g(n) := \lfloor n^{1/r} \rfloor$. Then the function R_g^{reg} is Ackermannian.*

PROOF. Given a real number $r > 0$ let $t := \lceil r \rceil$. We first observe that the function $k \mapsto \frac{k^{1/2t}}{2}$ grows eventually faster than the function $k \mapsto k^{1/4t}$ and therefore, by Claim 2.13, $k \mapsto (f_{g_t})_k(k)$ is Ackermannian. Set $\mu_{g_t}(k) := 4^t k^{2t}$. By Corollary 2.25, $R_{g_t}^{\mathrm{reg}}$ is Ackermannian. Therefore R_g^{reg} is Ackermannian, too. □

We conclude with a single primitive recursive procedure for coloring all of $[\mathbb{N}]^2$ whose Ramsey function is Ackermannian.

Theorem 2.27 *Suppose $g(n) = \lfloor n^{1/\operatorname{Ack}^{-1}(n)} \rfloor$ for $n > 0$ and $g(0) = 0$. There exists a g-regressive, primitive recursive coloring $C : [\mathbb{N}]^2 \to \mathbb{N}$ such that for every primitive recursive function $f : \mathbb{N} \to \mathbb{N}$ there exists $N_f \in \mathbb{N}$ such that for all $m > N_f$ and $H \subseteq m$ which is min-homogeneous for C it holds that $f(|H|) < m$.*

PROOF. We define a g-regressive coloring C by dividing $\mathbb{N}^+$ into disjoint intervals of the form $(\mu_{t-1}, \mu_t]$, defining a g-regressive coloring C_t for all pairs over each such interval. For each t, we specify an upper bound k_t on the sizes of C_t-min-homogeneous subsets of $(\mu_{t-1}, \mu_t]$. For the first interval we fix an ad-hoc coloring and for all other intervals we use the definition of C_g as described above. Finally, we integrate all colorings to a single coloring of all pairs over $\mathbb{N}$, by simply setting $C(m,n) = 0$ for m, n from different intervals and $C(0,n) = 0$ for all $n \in \mathbb{N}^+$. For notational convenience we start with $\mu_2 := 0$. We set $\mu_3 := 2^{61}$ and $\mu_t = \text{Ack}(t)$ for $t \geq 4$.

On $(\mu_2 = 0, \mu_3]$ fix C_3 as follows. Since $g(n) \geq 1$ for all $n > 0$ we may color pairs from $(0, 2^{61}]$ g-regressively by 2 colors. Using a simple probabilistic argument it may be shown that for any $k \geq 4$, there exists a 2-coloring of $\left[2^{k/2}\right]^2$ with no min-homogeneous set of size k. We set $k_3 := 122$ and let C_3 be a restriction of such a coloring to $(0, 2^{61}]$.

Now we need to define C_t for all $t > 3$. Let $k_4 := 98$ and $k_t := 16t^2 + 9t + 2$ for all $t > 4$. We color pairs over the interval $[\mu_{t-1}, (f_{g_t})_{k_t}(\mu_{t-1}))$ by C_g as defined above (Definition 2.20), using as parameters, $g := g_t$, as defined in Definition 2.14, and $k := k_t$. For formality, we fix the function $\mu_{g_t}(k) := \mu_{t-1}$ iff t is the least number such that $3 < t$ and $k \leq k_t$. For our needs, however, it suffices to observe that for all $t > 3$ it holds that $k_t \leq \frac{\sqrt{g_t(\mu_{g_t}(k_t))}}{2}$, which can easily be verified. We set C_t, for $t > 3$, to be the restriction of C_{g_t} to $(\mu_{t-1}, \mu_t]$ (See Claim 2.28 to observe that it is a restriction).

The following claim shows that the union of all intervals, indeed covers all $\mathbb{N}$.

Claim 2.28 $\text{Ack}(t) < (f_{g_t})_{k_t}(\mu_{t-1}))$ *for all* $t > 3$.

PROOF. We first prove Claim 2.28 for $t = 4$. Note that $k_4 = 98$. Observe that $\frac{61}{8} - 1 > \frac{61}{10}$ and hence for all $n \geq 2^{61}$ and for every $i \in \mathbb{N}$ it holds that $(f_{g_4})_i(n) \geq (f_{10})_i(n)$. By Claim 2.5 we know that $(f_{10})_{97}(2^{61}) > (\lfloor (2^{61})^{1/10} \rfloor)^{96} > 2^{576}$. Using the same argument again we also know that $(f_{10})_{97}(2^{576}) > 2^{5472}$. Thus, $(f_{g_4})_{k_5}(\mu_4) = (f_{g_4})_{98}(2^{61})$ is much larger than

$$(f_{g_4})^{(3)}_{97}(2^{61}) > (f_{g_4})_{97}((f_{10})^{(2)}_{97}(2^{61})) > (f_{g_4})_{97}(2^{5472})$$

Now, since $\frac{5472}{8} - 1 > \frac{5472}{9}$ it holds that for all $m \geq 2^{5472}$ and for every $i \in \mathbb{N}$ it holds that $(f_{g_4})_i(m) \geq (f_9)_i(m)$. Hence we have $(f_{g_4})_{97}(2^{5472}) \geq (f_9)_{97}(2^{5472})$ and by Claim 2.9, we have that $(f_9)_{97}(2^{5472}) > (f_9)_{9+8^2+24}(5^{2^8}) > (A_9(5))^{2^8}$ and thus, obviously larger than $A_4(4)$.

Now, let $t > 4$. Observe that $\mu_{t-1} > A_4(t-1)$ and therefore larger than 2^{4t}. Since for all $n \geq 2^{4t}$ and for every $i \in \mathbb{N}$ it holds that $(f_{g_t})_i(n) \geq (f_{4t})_i(n)$, we have that $(f_{g_t})_{k_t}(\mu_{t-1}) \geq (f_{4t})_{k_t}(\mu_{t-1})$. It also holds that $\mu_{t-1} > (4t)^{4t}$. Hence, by Claim 2.12 $(f_{4t})_{k_t}(\mu_{t-1}) > A_{k_t - 16t^2 - 8t - 2}(\mu_{t-1}) = A_t(Ack(t-1))$ which is obviously larger than $A_t(t)$ □

Finally, we define C as follows.

$$C(m,n) := \begin{cases} C_t(m,n) & \text{if } 0 < m,n \in (\mu_{t-1}, \mu_t], \\ 0 & \text{otherwise.} \end{cases}$$

Claim 2.29 *The coloring C is g-regressive.*

PROOF. Let $m,n \in \mathbb{N}$ be such that $m < n$. If $C(m,n) = 0$ then we have $C(m,n) \leq g(m)$. Otherwise, m and n are in the same interval. If $m,n \in (\mu_2, \mu_3]$ then $C(m,n) \leq 1 \leq \lfloor m^{1/Ack^{-1}(m)} \rfloor$ by definition of C_3. If $m,n \in (\mu_{t-1}, \mu_t]$ for some $t > 3$, then we have $Ack^{-1}(m) = t$. We also know C_t is g_t regressive on that interval and thus $C(m,n) = C_t(m,n) \leq \lfloor m^{1/t} \rfloor = \lfloor m^{1/Ack^{-1}(m)} \rfloor$. □

Claim 2.30 *The coloring C is primitive recursive.*

PROOF. It is primitive recursive to compute for an input n the last value of Ack below n. Thus, given input m,n one can determine whether there is some $t \geq 3$ so that $m,n \in (\mu_{t-1}, \mu_t]$. The computation of C on each $(\mu_{t-1}, \mu_t]$ is uniform and primitive recursive. So altogether, C is primitive recursive. □

Claim 2.31 *For any given $N \in \mathbb{N}$ with $\mathrm{Ack}^{-1}(N) < j$ for some $j > 3$, there is no C-min-homogeneous $H \subseteq [N]$ of size $(k_j)^2 + 123$.*

PROOF. Clearly, for all $t > 3$ it holds that $k_t < k_{t+1}$ and that $k_t > t$. Thus, since at any interval $(\mu_{t-1}, \mu_t]$ for $3 < t \leq j$, the largest min-homogeneous subset may be of size k_t and hence, no more than k_j. Therefore, in the union of all those intervals there is no min-homogeneous subset larger than $k_j(j-3) < (k_j)^2$. Now, in the first interval there can be no min-homogeneous of size 122. Thus, as we allow 0 to be an element of any min-homogeneous subset, so there is no min-homogeneous $H \subseteq [N]$ of size $(k_j)^2 + 123$ in the union of all intervals before $\mathrm{Ack}(j)$, of which $[N]$ is a subset.

To conclude the proof of Theorem 2.27, fix $f_1(i) = i^2 + 123$ and $f_2(i) = 16i^2 + 9i + 2$. Now, given some primitive recursive function f, let f' be some increasing primitive recursive function which bounds f. Note that the composition $h := f' \circ (f_1 \circ f_2)$ is also primitive recursive. Let $t_0 > 4$ be the least natural number such that for all $t \geq t_0$ it holds that $\mathrm{Ack}(t-1) > h(t)$. Let $N_f := \mathrm{Ack}(t_0)$. Given $m > N_f$ such that $m \in (\mu_{t-1}, \mu_t]$ and $H \subseteq m$ which is min-homogeneous for C, by Claim 2.31 we know that $|H| < k_t^2 + 123 = f_1(f_2(t))$. By monotonicity of f', we have $f'(|H|) < f'(f_1(f_2(t)))) = h(t)$. Since $N_f < m$ and by monotonicity of Ack, we have $t \geq t_0$ and thus $h(t) < \mathrm{Ack}(t-1) < m$. Now, $f(i) \leq f'(i)$ for all $i \in \mathbb{N}$ and therefore $f(|H|) \leq f'(|H|) < m$

This completes the proof of Theorem 2.27. $\square$

2.3 The Id-regressive Ramsey number of 82 *is larger than* $A_{53}(2^{2^{274}})$

We provide now a (huge) lower estimate on an Id-regressive Ramsey number for a reasonably small $k = 82$. The point to stress is that the bad colorings we had above work not only asymptotically but may be used to estimate small values. For more on small regressive Ramsey numbers see Blanchard [2].

Claim 2.32 *For $g = Id$ it holds that $R_g^{\mathrm{reg}}(82) > A_{53}(2^{2^{274}})$.*

PROOF. Let $\mu = 2^{14}$ and $k = 64$. By Claims 2.22 and 2.23 we know that there is a g-regressive coloring C_{Id} on the interval $[\mu, (f_{\mathrm{Id}})_k(\mu))$ which yields no $H \subseteq [\mu, (f_{\mathrm{Id}})_k(\mu))$ of size $k+1$ which is min-homogeneous for C_{Id}. Let us now examine the magnitude of $(f_{\mathrm{Id}})_k(\mu)$. By definition

$$(f_{\mathrm{Id}})_k(\mu) = (f_{g_1})_{64}(2^{14}) = (f_{g_1})_{63}^{(64)}(2^{14}).$$

Since for all $x > 2^6$ it holds that $\frac{x^{1/2}}{2} > x^{1/3}$ and by monotonicity, we may look at $(f_3)_{63}^{(64)}(2^{14})$ which, by Claim 2.5, is larger than $(f_3)_{63}^{(63)}((\lfloor 2^{14/3} \rfloor)^{62}) > (f_3)_{63}^{(63)}(2^{285})$. By applying the same argument again we get $(f_3)_{63}^{(63)}(2^{285}) > (f_3)_{63}^{(62)}(2^{5889})$. We go on applying Claim 2.5 in the straightforward manner until we establish that the latter term is larger than $(f_3)_{63}^{(59)}(2^{51981110})$ and then we start using 60 instead of 62 at the exponent which enables us to lose the rounding operation. Thus, we know that $(f_3)_{63}^{(59)}(2^{51981110}) > (f_3)_{63}^{(1)}(2^{51981110 * 20^{58}}) > (f_3)_{63}(2^{2^{276}})$. By applying Claim 2.9 to the latter term we get $(f_3)_{63}(2^{2^{276}}) = (f_3)_{53+2^2+6}((2^{2^{274}})^{2^2}) > (A_{53}(2^{2^{274}}))^{2^2}$ which is obviously larger than $A_{53}(2^{2^{274}})$.

On $[0, 13)$ there is an Id-regressive coloring with no min-homogeneous set with more than 4 elements (see [2]). On $[13, 2^{14})$ let $C(m, n)$ be the largest position of a different digit in the base 2 expansions of m and n. This coloring is Id-regressive, since $C(m, n) \leq 13$ for all such m, n and admits no min-homogeneous set of size 14. Coloring m, n from different intervals by 0 produces then a coloring on the interval $[0, A_{53}(2^{2^{274}}))$ with no min-homogeneous set of size larger than $4 + 13 + 64 = 81$. $\square$

3 The Phase Transition of g-large Ramsey numbers.

We prove now that the threshold for Ackermannian g-large Ramsey numbers lies above all functions $\log(n)/f^{-1}(n)$ obtained from an increasing primitive recursive f and below the function $\log(n)/\operatorname{Ack}^{-1}(n)$.

Worded differently, for a nondecreasing and unbounded g to have primitive recursive g-large Ramsey numbers it is necessary and sufficient that g is eventually dominated by $\log(n)/t$ for all $t > 0$ and that the rate at which g gets below $\log(n)/t$ is not too slow, namely, is primitive recursive in t: if g gets below $\log(n)/t$ only after an Ackermannianly long time $M(t)$, then the g-large Ramsey numbers are still Ackermannian.

In this section we shall work with a new hierarchy of functions F_m. It is similar to that of A_m, only it starts with a faster growing function than the successor function:

$$F_m(i) := \begin{cases} 2^i & \text{if } m = 0, \\ F_{m-1}^{(i \dot{-} 1)}(i) & \text{otherwise.} \end{cases}$$

Here $i \dot{-} 1 = i - 1$ if $i > 0$ and 0 otherwise. This is merely done for technical convenience and helps us handle the logarithm much better. For any $m \in \mathbb{N}$, F_m is an increasing primitive recursive function. The function $F\colon \mathbb{N} \to \mathbb{N}$, defined by $F(i) := F_i(i)$, is Ackermannian. In fact, F and Ack have almost the same growth rate.

We employ classical bounds by Erdős and Rado for the lower bound and a result by Abbott [1] for the upper bound which relies on the probabilistic method of Erdős. The following lemma follows e.g. from Theorem 1 in [6].

Lemma 3.1 $R(k, c) \leq c^{c \cdot k - 1} = 2^{(c \cdot k - 1) \cdot \log(c)}$ *for any* c, $k \geq 2$.

For $m \in \mathbb{N}$ and a function $B : \mathbb{N} \to \mathbb{N}$ set

$$f_m(i) := \left\lfloor \frac{\log(i)}{F_m^{-1}(i)} \right\rfloor \quad \text{and} \quad f_B(i) := \left\lfloor \frac{\log(i)}{B^{-1}(i)} \right\rfloor .$$

Lemma 3.2 *Let* $B : \mathbb{N} \to \mathbb{N}$ *be a nondecreasing and unbounded positive function. Then for every* $c, k \in \mathbb{N}$ *it holds that* $R(t, c) + B(c \cdot \lceil \log(c) \rceil) \to^*_{f_B} (k)_c$, *where* $t := \max\{k, B(c \cdot \lceil \log(c) \rceil)\}$.

PROOF. Given c, k let $N := R(t, c)$. By Lemma 3.1

$$N + B(c \cdot \lceil \log(c) \rceil) \;\leq\; 2^{(c \cdot t - 1) \cdot \log(c)} + B(c \cdot \lceil \log(c) \rceil) \;\leq\; 2^{c \cdot \log(c) \cdot t} .$$

Now, let $C\colon [N + B(c \cdot \lceil \log(c) \rceil)]^2 \to c$ be given. Since $N + B(c \cdot \lceil \log(c) \rceil) - B(c \cdot \lceil \log(c) \rceil) = N$, there is an $H \subseteq [B(c \cdot \lceil \log(c) \rceil), N + B(c \cdot \lceil \log(c) \rceil))$

homogeneous for C, such that $|H| \geq t$. Therefore, we have

$$\frac{\log(\min(H))}{B^{-1}(\min(H))} \leq \frac{c \cdot \log(c) \cdot t}{B^{-1}(B(c \cdot \lceil \log(c) \rceil))} \leq t \leq |H| .$$

Thus H is f_B-large. $\square$

Theorem 3.3 *For every fixed m the function $R^*_{f_m}$ is primitive recursive.*

PROOF. By Lemma 3.2, $R^*_{f_m}$ is bounded by a primitive recursive function and thus is itself primitive recursive, as the class of primitive recursive functions is closed under the bounded μ-operator. $\square$

Now we turn to the counterpart of Theorem 3.3.

Lemma 3.4 *There are positive integers c_0 and M such that for all $c \geq 2$ and $k \geq M$ it holds that $R(k, c) \geq 2^{\frac{1}{c_0} \cdot c \cdot k}$.*

PROOF. See Abott [1]. $\square$

Lemma 3.5 *Let $M \geq 2$ and c_0 be the constants from Lemma 3.4. Let $d \geq 4$ be arbitrary, but fixed. Put $\varepsilon := \frac{1}{d}$ and $K := 2 \cdot d \cdot M + 1$. Then*

$$R^*_{\hat{f}_\varepsilon}(k, c_0 \cdot M \cdot d \cdot 2) \geq 2^{2^{k \cdot d}}$$

for all $k \geq K$, where $\hat{f}_\varepsilon(i) := \varepsilon \cdot \log(i)$.

PROOF. Pick $k \geq K$. Let $n_0 := 0$, $n_1 := R(k, c_0) - 1$, and for $1 \leq i < k - 1$

$$n_{i+1} := n_i + R(\lfloor \varepsilon \cdot \log(n_i) \rfloor, c_0 \cdot M \cdot d \cdot 2 - 1) - 1.$$

Finally put $n := n_{k-1}$. We claim:

$$n \not\rightarrow^*_{\hat{f}_\varepsilon} (k)_{c_0 \cdot M \cdot d \cdot 2}$$

Choose $C_0 : [n_0, n_1)^2 \to c_0$ such that every C_0-homogeneous $H \subseteq [n_0, n_1)$ satisfies $|H| < k$. For $1 \leq i < k - 1$ choose

$$C_i : [n_i, n_{i+1})^2 \to c_0 \cdot M \cdot d \cdot 2 - 1$$

such that if H is C_i-homogeneous then $|H| < \lfloor \varepsilon \cdot \log(n_i) \rfloor$.

Define $C\colon [n]^2 \to c_0 \cdot M \cdot d \cdot 2$ as follows:

$$C(u,v) := \begin{cases} C_i(u,v)+1 & \text{if } n_i \le u < v < n_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let H be C-homogeneous. If the color of H is 0 then $|(H \cap [n_i, n_{i+1}))| \le 1$, hence $|H| \le k-1 < k$. If the color of H under C is greater than 0 then $H \subseteq [n_j, n_{j+1}]$ for some j and H is homogeneous for C_j. If $j=0$ then $|H| < k$ by choice of C_0. If $j > 0$ then

$$|H| < \lfloor \varepsilon \cdot \log(n_j) \rfloor \le \lfloor \varepsilon \cdot \log(\min(H)) \rfloor .$$

This implies that $n < R^*_{\hat{f}_\varepsilon}(k, c_0 \cdot M \cdot d \cdot 2)$.

Now we use induction on $1 \le i < k$ to prove that $n_i \ge 2^{2^i \cdot d \cdot M}$. For $i = 1$ we have, by Lemma 3.4,

$$n_1 \ge 2^{\frac{1}{c_0} \cdot k \cdot c_0} - 1 \ge 2^{2 \cdot d \cdot M}$$

since $k \ge K = 2 \cdot d \cdot M + 1$. The induction hypothesis yields for $i < k-1$

$$\lfloor \varepsilon \cdot \log(n_i) \rfloor \ge \lfloor \varepsilon \cdot 2^i \cdot d \cdot M \rfloor = 2^i \cdot M .$$

Thus by Lemma 3.4 we have for the induction step

$$n_{i+1} \ge R(\lfloor \varepsilon \cdot \log(n_i) \rfloor, c_0 \cdot M \cdot d \cdot 2 - 1) \ge 2^{\frac{1}{c_0} \cdot (c_0 \cdot M \cdot d \cdot 2 - 1) \cdot 2^i \cdot M} \ge 2^{2^{i+1} \cdot d \cdot M}$$

since $M \ge 2$. Hence $R^*_{\hat{f}_\varepsilon}(k, c_0 \cdot M \cdot d \cdot 2) > n = n_{k-1} \ge 2^{2^{k-1} \cdot d \cdot M} \ge 2^{2^k \cdot d}$. □

Lemma 3.6 *With the notation of Lemma 3.5 we have:*

$$R^*_{\hat{f}_\varepsilon}(k, c_0 \cdot d \cdot M \cdot 2 + m) \ge 2^{d \cdot F_m(k)}$$

PROOF. We show the claim by induction on m. If $m = 0$, it is simply Lemma 3.5, since $F_0(k) = 2^k$.

Now assume that the claim is true for $m \ge 0$. Put $n_0 := 1$ and $n_1 := n_0 + R^*_{\hat{f}_\varepsilon}(k, c_0 \cdot d \cdot M \cdot 2 + m) - 1 = R^*_{\hat{f}_\varepsilon}(k, c_0 \cdot d \cdot M \cdot 2 + m)$. By recursion on $i > 0$ define

$$n_{i+1} := R^*_{\hat{f}_\varepsilon}(\lfloor \varepsilon \cdot \log(n_i) \rfloor, c_0 \cdot d \cdot M \cdot 2 + m) - 1.$$

Finally put $n := n_{k-1}$. We claim that

$$[1, n) \nrightarrow^*_{\hat{f}_\varepsilon} (k)_{c_0 \cdot d \cdot M \cdot 2 + m + 1}.$$

Choose $C_0 : [n_0, n_1)^2 \to c_0 \cdot d \cdot M \cdot 2 + m$ such that every C_0-homogeneous H satisfies $|H| < \max\{k, \hat{f}_\varepsilon(\min H)\}$. And for each $1 \le i < k-1$ choose

$C_i : [n_0, n_{i+1})^2 \to c_0 \cdot d \cdot M \cdot 2 + m$ such that every C_i-homogeneous $H \subseteq [n_0, n_{i+1})$ satisfies $|H| < \max\{\lfloor \varepsilon \cdot \log(n_i) \rfloor, \hat{f}_\varepsilon(\min H)\}$.

Define $C \colon [1, n)^2 \to c_0 \cdot M \cdot d \cdot 2 + m + 1$ as follows:

$$C(u, v) := \begin{cases} C_i(u, v) + 1 & \text{if } n_i \le u < v < n_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let H be C-homogeneous. If the color of H is 0 then $|H \cap [n_i, n_{i+1})| \le 1$, hence $|H| \le k - 1 < k$. If the color of H is greater than 0 then $H \subseteq [n_j, n_{j+1})$ for some j and H is homogeneous for C_j. If $j = 0$ then $|H| < \max\{k, \hat{f}_\varepsilon(\min H)\}$. If $j > 0$ then

$$|H| < \max\{\varepsilon \cdot \log(n_j), \hat{f}_\varepsilon(\min H)\} \le \hat{f}_\varepsilon(\min H)\,.$$

By induction on $1 \le i < k$ we show $n_i \ge 2^{d \cdot F_m^{(i)}(k)}$. Indeed, for the induction start we have

$$n_1 = R^*_{\hat{f}_\varepsilon}(k, c_0 \cdot d \cdot M \cdot 2 + m) \ge 2^{d \cdot F_m^{(1)}(k)}$$

by the main induction hypothesis. The induction hypothesis yields for $i \ge 1$ that $\varepsilon \cdot \log(n_i) \ge F_m^{(i)}(k) \ge k \ge K$. Hence the main induction hypothesis yields

$$n_{i+1} = R^*_{\hat{f}_\varepsilon}(\lfloor \varepsilon \cdot \log(n_i) \rfloor, c_0 \cdot d \cdot M \cdot 2 + m) \ge 2^{d \cdot F_m(F_m^{(i)}(k))} = 2^{d \cdot F_m^{(i+1)}(k)}$$

Therefore $n = n_{k-1} \ge 2^{d \cdot F_m^{(k-1)}(k)} = 2^{d \cdot F_{m+1}(k)}$. The assertion follows. $\square$

Theorem 3.7 *Suppose $B : \mathbb{N} \to \mathbb{N}$ is nonzero, nondecreasing and unbounded, and $B(i) \le F(i)$ for all i. Let c_0 and M be as in Lemma 3.4. Then*

$$N(d) := R^*_{f_B}(2 \cdot d \cdot M + 1, c_0 \cdot d \cdot M \cdot 2 + d) > B(d)$$

for all $d \ge 4$.

PROOF. Assume to the contrary that it is not so for some $d \ge 4$. Then for any $i \le N(d)$ we have

$$\frac{\log(i)}{B^{-1}(i)} \ge \frac{1}{d} \cdot \log(i)$$

since $B^{-1}(i) \le d$. Set $K_d := 2 \cdot d \cdot M$, $\varepsilon := \frac{1}{d}$, and denote $\hat{f}_\varepsilon(i) = \varepsilon \cdot \log(i)$. Clearly, every g_B-large set for a given coloring $C : [N(d)]^2 \to c_0 \cdot K_d + d$ is also a $\hat{f}_\varepsilon$-large set for C. Thus, we have

$$\begin{aligned} R^*_{f_B}(K_d + 1, c_0 \cdot K_d + d) &\ge R^*_{\hat{f}_\varepsilon}(K_d + 1, c_0 \cdot K_d + d) \\ &\ge F_d(K_d) \\ &> F(d) \\ &\ge B(d) \end{aligned}$$

by Lemma 3.6. Contradiction! $\square$

Theorem 3.8 *Suppose $B : \mathbb{N} \to \mathbb{N}$ is positive, unbounded and nondecreasing. Then the function $R^*_{f_B}$ is Ackermannian iff B is Ackermannian.*

PROOF. Suppose B is Ackermannian. By replacing B with $\min\{B, F\}$, we assume that $B(i) \leq F(i)$ for all $i \in \mathbb{N}$. This is done with no loss of generality, since clearly, if $B'(i) \leq B(i)$ for all $i \in \mathbb{N}$ and $R^*_{f_{B'}}$ is Ackermannian, then $R^*_{f_B}$ is Ackermannian too.

By the previous theorem, $R^*_{f_B}$ composed with the primitive recursive functions $r_1(i) := 2 \cdot i \cdot M + 1$ and $r_1(i) := c_0 \cdot i \cdot M \cdot 2 + i$ is Ackermannian. Therefore, $R^*_{f_B}$ itself is Ackermannian.

Conversely, suppose that B is not Ackermannian, and fix an increasing primitive recursive function f so that for infinitely many $i \in \mathbb{N}$ it holds that $B(i) < f(i)$. For each such i, let $c_i := \max\{c : c \cdot \log(c) \leq i\}$ and let $k_i := B(i)$. By Lemma 3.2, it holds that $R(k_i, c_i) + B(c_i \cdot \lceil \log(c_i) \rceil) \to^*_{f_B} (k_i)_{c_i}$. Since $f(i) \geq B(c_i \cdot \lceil \log(c_i) \rceil)$, it holds that $R^*_{f_B}(k_i, c_i) \leq R(k_i, c_i) + f(i)$. This is true for infinitely many c_i and infinitely many k_i. Thus, $R^*_{f_B}$ is not Ackermannian. $\square$

4 Phase transition from homogeneous to min-homogeneous Ramsey Numbers

We look now at the threshold g at which one can guarantee the usual Ramsey theorem for g-regressive colorings, that is, have homogeneous rather than just min-homogeneous sets.

Theorem 4.1 *Suppose $f : \mathbb{N} \to \mathbb{N}^+$ is nondecreasing and unbounded, and let $g(x) = \lfloor \frac{\log(x)}{f(x)\log(\log(x))} \rfloor$ for $x \geq 4$ and $g(x) = 0$ for $x < 4$. Then for all k there exists some N so that $N \to (k)_g$.*

PROOF. Given $k \geq 4$, find N_1 such that $f(N_1) > k$. Observe that for all $N_1 \leq m_1 \leq m_2$, it holds that $g(m_1) \leq \lfloor \frac{\log(m_2)}{k\log(\log(m_2))} \rfloor$. This is because the function $\frac{z}{\log z}$ is not decreasing for $z \geq 2$. Let $N := \max\{2N_1, 2^{2^k}\}$. Clearly, $\lfloor \frac{\log(N)}{k\log(\log(N))} \rfloor \geq 1$. We claim that any g-regressive function defined on $[N]^2$ admits a k sized homogeneous set.

Let $C : [N]^2 \to \mathbb{N}$ be g-regressive and $C' : [N_1, N]^2 \to c$ be its restriction, where $c := \lfloor \frac{\log(N)}{k\log(\log(N))} \rfloor + 1$. Note that we have, since $k \leq \log(\log(N))$,

$$
\begin{aligned}
c^{c\cdot k} &\leq \left(\frac{2\log(N)}{k\log(\log(N))}\right)^{k\left(\frac{\log(N)}{k\log(\log(N))}+1\right)}\\
&\leq N\cdot\frac{(\log(N))^k}{N^{\log(\log(\log(N)))/\log(\log(N))}}\cdot\frac{1}{\log(\log(N))}\\
&< \frac{N}{\log(\log(N))} < \frac{N}{2} \leq N-N_1
\end{aligned}
$$

By the standard Ramsey Theorem, there is a k sized C'-homogeneous set $H\subseteq[N_1,N]$. Hence C admits a k sized homogeneous set.

It should be noted that this is of interest when f grows slowly (e.g. $f(m)=\log^*(m)$). □

Theorem 4.2 *Suppose $j\in\mathbb{N}$ and $g(i)=\frac{\log(i)}{j}$. Then there exists some k such that for all N it holds that $N\not\rightarrow(k)_g$*

PROOF. Given $j\geq 2$ we set $s:=2^j$ and $k:=2s+1$ and construct a g-regressive coloring $C:\mathbb{N}^2\to\mathbb{N}$ where there exists no $H\subseteq\mathbb{N}$ of size $\geq k$ that is homogeneous for C. For any $n\in\mathbb{N}$, let $r_s(n):=(n_0,\dots,n_{\ell-1})$, where $\ell:=\lfloor\log_s(n)\rfloor+1$ and $n_i<s$, be the representation of n in s basis, i.e. $n=n_0\cdot s^{\ell-1}+\cdots+n_{\ell-1}\cdot s^0$.

For any $m,n\in\mathbb{N}$ such that $m<n$ and $\ell=\lfloor\log_s(m)\rfloor+1=\lfloor\log_s(n)\rfloor+1$, let $f(m,n):=\min\{i<\ell : m_i<n_i\}$, where $r_s(m)=(m_0,\dots,m_{\ell-1})$ and $r_s(n)=(n_0,\dots,n_{\ell-1})$. We define C as

$$
C(m,n)=\begin{cases}\lfloor\log_s(m)\rfloor & \text{if } \lfloor\log_s(m)\rfloor\neq\lfloor\log_s(n)\rfloor;\\ f(m,n) & \text{if } \lfloor\log_s(m)\rfloor=\lfloor\log_s(n)\rfloor.\end{cases}
$$

Note that C is g-regressive since for all $m,n\in\mathbb{N}$ it holds that $C(m,n)\leq\log_s(m)=\frac{\log(m)}{j}$.

Observation 4.3 *Let $Y=\{y_1,y_2,...,y_{s+1}\}$ where $y_1<y_2<...<y_{s+1}$, be a homogeneous set for C. Then $\lfloor\log_s(y_1)\rfloor<\lfloor\log_s(y_{s+1})\rfloor$.*

To show Observation 4.3, let Y be a homogeneous set for C and suppose to the contrary that $\lfloor\log_s(y_1)\rfloor=\lfloor\log_s(y_{s+1})\rfloor$. From the definition of C we get that f is constant on Y. Thus elements of Y, pairwise differ in the i'th value in their s basis representation for some index i, which is impossible since there are only s possible values for any index. Contradiction.

Now let $H=\{x_1,x_2,...,x_{2s+1}\}$, where $x_1<x_2<...<x_{2s+1}$, and suppose to the contrary that H is homogeneous for C. By observation 4.3 we get

that $\lfloor \log_s(x_1) \rfloor < \lfloor \log_s(x_{s+1}) \rfloor < \lfloor \log_s(x_{2s+1}) \rfloor$ and therefore $C(x_1, x_{s+1}) < C(x_{s+1}, x_{2s+1})$ contrary to homogeneity. $\square$

5 Conclusion

We have proved sharp phase transition thresholds for the regressive and Paris-Harrington Ramsey numbers. Although the proofs for these results are quite different it might be interesting to see that they can be motivated by a unifying underlying phase transition principle. As it turned out, finite combinatorics provides bounds (on finite Ramsey numbers) which also provide good bounds on regressive and Paris Harrington Ramsey numbers below the threshold. Indeed these calculations provide a priori guesses where the desired thresholds might be located. In our examples it turned out that the guesses were good since for parameter functions growing faster than the threshold function, a suitable iteration argument shows that the induced Ramsey functions have extraordinary growth. In vague analogy with dynamical systems one might consider the threshold region as an unstable fixed point of a renormalization operator given by the bounds on finitary Ramsey numbers.

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